3-MANIFOLDS, TANGLES AND PERSISTENT INVARIANTS

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ABSTRACT: Given a compact, connected, oriented 3-manifold M with boundary, and epimorphism χ from $H_1 M$ to a free abelian group Π, two invariants β, $\tau \in \mathbb{Z}\Pi$ are defined. If M embeds in another such 3-manifold N such that χ_N factors through χ, then the product $\beta\tau$ divides $\Delta_0(H_1\tilde{N})$.

A theorem of D. Krebes concerning 4-tangles embedded in links arises as a special case. Algebraic and skein theoretic generalizations for $2n$-tangles provide invariants that persist in the corresponding invariants of links in which they embed. An example is given of a virtual 4-tangle for which Krebes's theorem does not hold.

Keywords: Tangle, virtual tangle, link, branched cover, determinant

1. Introduction.

Suppose that a $2n$-tangle t embeds in a link ℓ. It is natural to ask which invariants of t necessarily persist as invariants of ℓ. In [**Kr99**] D. Krebes considered the case that t is a 4-tangle. There he proved that any positive integer dividing the determinants of both the numerator closure and the denominator closure of the tangle also divides the determinant of ℓ.

Krebes's argument is a blend of combinatorics and topology. In [**Ru00**] D. Ruberman gave another proof of Krebes's theorem using purely topological techniques. Ruberman exploited a well-known relationship between the determinant of a link and the first homology of its 2-fold cyclic branched cover. From his perspective Krebes's theorem is a result about invariants of compact, oriented 3-manifolds that persist as invariants of rational homology 3-spheres in which they embed.

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We extend Ruberman's methods in order to prove a generalization of Krebes's theorem for $2n$-tangles (see Theorem 2.5). Given a $2n$-tangle t we define two invariants $\tau, \beta \in \mathbb{Z}\Pi$, where Π is a free abelian group. The rank of Π, denoted by d, depends on the category in which we work: If we color and orient the components, then d can be chosen to be the number of components of t. Then whenever t embeds in an oriented link ℓ such that distinct components of t lie in different components of ℓ, the product $\tau\beta$ divides the multivariable Alexander polynomial of ℓ. At the other extreme, we may choose to ignore both the colors and orientations of t. In that case $d = 0$ and τ, β are integers. If t embeds in an unoriented link ℓ, then the product $\tau\beta$ divides the determinant of ℓ. The latter statment is seen to be Krebes's theorem for 4-tangles.

The second and third authors gave a short, elementary proof of Krebes's theorem when the divisor is prime. The proof immediately extends when the divisor is square-free. The argument, based on the combinatorial technique of Fox coloring, holds in the larger category of virtual tangles and links [**SW99**]. In Section 4 we consider the virtual category. We give an example that shows that Krebes's theorem is not valid in the larger category unless the divisor is square-free.

Another generalization of Krebes's theorem for $2n$-tangles, proved using Kauffman bracket skein theory and Temperley-Lieb algebra, is in [**KSW00**]. We extend this approach to other skein theories in the last section.

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2. Persistent invariants of submanifolds. Let Π be a free abelian multiplicative group on $d \geq 0$ generators x_i. The group ring $\Lambda = \mathbb{Z}\Pi$ is a Noetherian unique factorization domain with automorphism $r \mapsto \bar{r}$ extending the assignment $x_i \mapsto x_i^{-1}$, for all i.

Let H be a finitely generated Λ-module. The *rank* of H is the dimension of the vector space $Q(\Lambda) \otimes_\Lambda H$, where $Q(\Lambda)$ is the field of fractions of Λ. The Λ-*torsion submodule* of H is $TH = \{a \in H \mid ra = 0 \text{ for some nonzero } r \in \Lambda\}$. We denote the *Betti module* H/TH by BH. Assume that we have a presentation

$$\Lambda^p \xrightarrow{R} \Lambda^q \to H \to 0.$$

By adding trivial relators, if necessary, we can assume that $q \leq p$. For each $0 \leq k < q$, the *kth elementary divisor* $\Delta_k(H)$ is the greatest common divisor of the $(q-k) \times (q-k)$ subdeterminants of the matrix representing R. It is well defined up to multiplication by a unit in Λ. By convention, $\Delta_k(H) = 0$ if k is negative, while $\Delta_k(H) = 1$ if $k \geq q$. For each k, the polynomial $\Delta_k(H)$ is an invariant of H; in particular, it does not depend on the particular choice of matrix representing R.

Consider a compact 3-manifold X with boundary ∂X decomposed as the union of two surfaces $\partial_+ X$ and $\partial_- X$; if both are nonempty then their intersection should be a

1-manifold. Let $\chi : H_1X \to \Pi$ be an epimorphism. The map χ determines an abelian cover $p : \tilde{X} \to X$ with deck transformation group Π. We denote the preimage $p^{-1}(\partial_\pm X)$ by $\partial_\pm \tilde{X}$. The homology groups $H_*(\tilde{X}), H_*(\tilde{X}, \partial\tilde{X})$ (integer coefficients understood) are in fact finitely generated Λ-modules. We consider the composite homomorphism

$$\nabla : H_1\partial_+\tilde{X} \stackrel{i_{1+}}{\to} H_1\tilde{X} \stackrel{\pi}{\to} BH_1\tilde{X}, \tag{2.1}$$

where i_{1+} is the map induced by inclusion and π is the natural quotient map.

Definition 2.1. The *boundary invariant* $\beta(X, \partial_+X)$ is $\Delta_0(BH_1\tilde{X}/\mathrm{im}\ \nabla)$. The *torsion invariant* $\tau(X)$ is $\Delta_0(TH_1\tilde{X})$.

Although $\tau(X)$ and $\beta(X, \partial_+X)$ depend on χ, we omit specific mention of χ in our notation for the sake of convenience.

Let M, N be compact, connected oriented 3-manifolds with $M \subset N$. Regard N as the union of M and another compact, oriented 3-manifold M' with $M \cap M' = \partial_+M$. We assume that χ extends over H_1N. The preimage $p^{-1}(M)$ is connected and can be identified with $\tilde{M}$. Assume that if $\partial\tilde{N}$ is nonempty, then each component is noncompact; the assumption is equivalent to the statement that each component of ∂N contains a cycle z such that $\chi([z]) \neq 0$.

Theorem 2.2. Under the above hypotheses, $\beta(M, \partial_+M)\tau(M)$ divides $\Delta_0(H_1\tilde{N})$.

The proof of Theorem 2.2 when $d > 0$ (that is, when the covers are nontrivial) uses Blanchfield duality, which we review. Let X be a compact, connected n-manifold with boundary $X = \partial_+X \cup \partial_-X$. If $p : \tilde{X} \to X$ is any connected cover of X with covering group Π, there there are nondegenerate Λ-sesquilinear forms

$$BH_p(\tilde{X}, \partial_+\tilde{X}) \times BH_{n-p}(\tilde{X}, \partial_-\tilde{X}) \to \Lambda;$$
$$T_DH_p(\tilde{X}, \partial_+\tilde{X}) \times T_DH_{n-p-1}(\tilde{X}, \partial_-\tilde{X}) \to Q(\Lambda)/\Lambda.$$

Here T_DH denotes the quotient TH/DH, where $DH = \{a \in H \mid r_1a = \cdots = r_qa = 0,\ \text{for some coprime } r_1, \ldots, r_q \in \Lambda\ (q \geq 2)\}$. Details can be found in [**Bl57**] or [**Kw96**].

Lemma 2.3. (1) If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated Λ-modules, then $\Delta_0(B) \doteq \Delta_0(A)\Delta_0(C)$, where $\doteq$ denotes equality in Λ up to multiplication by a unit.

(2) Let H be a finitely generated Λ-module. Then H is a Λ-torsion module if and only if $\Delta_0(H) \neq 0$. More generally, if H is any finitely generated Λ-module of rank r, then

$$\Delta_k(H) \doteq \begin{cases} 0 & \text{for } k < r, \\ \Delta_{k-r}(TH) & \text{for } k \geq r \end{cases}.$$

(3) Let H be a finitely generated Λ-module. If D_0 is a submodule of DH, then $\Delta_k(H/D_0) = \Delta_k(H)$ for any $k \geq 0$.

(4) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of modules over any ring, then for any submodule $D \subset B$, the following sequence is also exact.

$$0 \to A/f^{-1}(D) \xrightarrow{\bar{f}} B/D \xrightarrow{\bar{g}} C/g(D) \to 0.$$

Proof. Lemma 2.3 (1) is well known. A proof can be found on page 92 of [**Kw96**], for example. The second and third statements of Lemma 2.3 are proved in [**Bl57**] (see Lemmas 4.3 and 4.10 for (2); for (3) see the proof of Theorem 4.7). The proof of statement (4) is routine and is left to the reader. ■

Proof of Theorem 2.2. If $\Delta_0(H_1\tilde{N}) = 0$, then the proof of (1) is trivial. Therefore we assume that $\Delta_0(H_1\tilde{N}) \neq 0$. By Lemma 2.3(2) $BH_1\tilde{N} = 0$. It follows by Blanchfield duality that $BH_2(\tilde{N}, \partial\tilde{N}) = 0$. Thus $H_2(\tilde{N}, \partial\tilde{N})$ is a Λ-torsion module.

By hypothesis, each component of $\partial\tilde{N}$ is noncompact. Hence $H_2\partial\tilde{N} = 0$. From the exact sequence of the pair $\partial\tilde{N} \subset \tilde{N}$:

$$\cdots \to H_2\partial\tilde{N} \to H_2\tilde{N} \to H_2(\tilde{N}, \partial\tilde{N}) \to \cdots$$

we see that $H_2\tilde{N}$ is also a Λ-torsion module.

By excision $H_*(\tilde{N}, \tilde{M}) \cong H_*(\tilde{M}', \partial_+\tilde{M})$. Hence Blanchfield duality pairs $T_D H_2(\tilde{N}, \tilde{M})$ with $T_D H_0(\tilde{M}', \partial_-\tilde{M})$. Since $H_0(\tilde{M}', \partial_-\tilde{M})$ is free, the module $T_D H_0(\tilde{M}', \partial_-\tilde{M})$ is trivial. Hence $TH_2(\tilde{N}, \tilde{M}) \cong DH_2(\tilde{N}, \tilde{M})$.

Consider now the exact sequence of the pair $\tilde{M} \subset \tilde{N}$:

$$\cdots \to TH_2\tilde{N} \xrightarrow{k_2} H_2(\tilde{N}, \tilde{M}) \xrightarrow{\partial_2} H_1\tilde{M} \xrightarrow{j_1} H_1\tilde{N} \to \cdots$$

From this, $H_1\tilde{M}/\text{im } \partial_2$ is isomorphic to a submodule of $H_1\tilde{N}$. Lemma 2.3 (1) implies that $\Delta_0(H_1\tilde{M}/\text{im } \partial_2)$ divides $\Delta_0(H_1\tilde{N})$. It also follows that $\text{im } \partial_2 \cap TH_1\tilde{M} \subset DH_1\tilde{M}$. The reason is the following. If $\partial_2 a \in TH_1\tilde{M}$, then $ra \in \ker \partial_2$, for some $0 \neq r \in \Lambda$. The exact sequence shows that $ra \in \text{im } k_2$ is a torsion element. Consequently a itself is torsion. Hence $a \in TH_2(\tilde{N}, \tilde{M}) \cong DH_2H(\tilde{N}, \tilde{M})$, and so $\partial_2 a \in DH_1\tilde{M}$.

Consider the canonical short exact sequence:

$$0 \to TH_1\tilde{M} \to H_1\tilde{M} \xrightarrow{\pi} BH_1\tilde{M} \to 0,$$

where the first map is inclusion and the second is the natural quotient projection. By Lemma 2.3 (4):

$$0 \to \frac{TH_1\tilde{M}}{\text{im } \partial_2 \cap TH_1\tilde{M}} \to \frac{H_1\tilde{M}}{\text{im } \partial_2} \xrightarrow{\pi} \frac{BH_1\tilde{M}}{\text{im } \pi\partial_2} \to 0.$$

By Lemma 2.3 (1) we have

$$\Delta_0\Big(\frac{H_1\tilde{M}}{\text{im } \partial_2}\Big) \doteq \Delta_0\Big(\frac{TH_1\tilde{M}}{\text{im } \partial_2 \cap TH_1\tilde{M}}\Big)\Delta_0\Big(\frac{BH_1\tilde{M}}{\text{im } \pi\partial_2}\Big).$$

Since $\text{im } \partial_2 \cap TH_1\tilde{M} \subset DH_1\tilde{M}$, by Lemma 2.3 (3)

$$\Delta_0\Big(\frac{TH_1\tilde{M}}{\text{im } \partial_2 \cap TH_1\tilde{M}}\Big) \doteq \Delta_0(TH_1\tilde{M}),$$

which is the torsion invariant $\tau(M)$. Also, $BH_1\tilde{M}/\text{im } \pi\, i_{1+}$ is a quotient of $BH_1\tilde{M}/\text{im } \pi\, \partial_2$, and hence the boundary invariant

$$\beta(M, \partial_+M) = \Delta_0\Big(\frac{BH_1\tilde{M}}{\text{im } \pi i_1}\Big) \text{ divides } \Delta_0\Big(\frac{BH_1\tilde{M}}{im\ \pi\partial_2}\Big),$$

again using Lemma 2.3 (1). Hence $\beta(M, \partial_+M)\tau(M)$ divides $\Delta_0(H_1\tilde{M}/\text{im } \pi\partial_2)$, which we have previously seen divides $\Delta_0(H_1\tilde{N})$. ∎

3. Application to tangles. A $2n$-*tangle*, for n a positive integer, consists of n disjoint arcs and any finite number of simple closed curves properly embedded in the 3-ball. Two $2n$-tangles are regarded as the same if one can be transformed into the other by an ambient isotopy of the 3-ball that fixes each point on the boundary. As usual we represent $2n$-tangles by diagrams. Two diagrams represent the same $2n$-tangle if one can be transformed into the other by a finite sequence of Reidemeister moves.

Definition 3.1. A $2n$-tangle t *embeds* in a link ℓ if some diagram for t extends to a diagram for ℓ.

A 4-tangle is called simply a *tangle*. By joining the top ends and then the bottom ends one obtains a link $n(t)$, the *numerator* of t. Joining the left-hand ends and then right-hand ends produces the *denominator* closure $d(t)$. See Figure 1.

The *determinant* $\det(\ell)$ can be defined in many ways. It is the absolute value of the one-variable Alexander polynomial (see below) of ℓ evaluated at -1. It is also the order of the first homology of the 2-fold cover of S^3 branched over ℓ, provided that the group is finite; if not, then the determinant is zero. See for example [**Li97**].

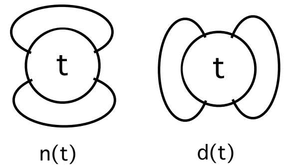

Figure 1. Diagrams of $n(t)$ and $d(t)$

Theorem 3.2. [**Kr99**] If a tangle t embeds in a link ℓ, then the greatest common divisor of $\det(n(t))$ and $\det(d(t))$ divides $\det(l)$.

Krebes's theorem generalizes in various ways. In order to state some of these, we need some terminology. If a $2n$-tangle or link has a specified direction for each component, then it is *oriented*. If it has a specified color for each component, then it is *colored*.

Let t be a colored, oriented $2n$-tangle with exterior $E_t = B^3 - int\ N(t)$. Here $N(t)$ denotes a tubular neighborhood of t. The homology group H_1E_t is freely generated by d oriented meridians, where d is the number of connected components of t.

We use Defintion 2.1 to associate invariants β, τ to the 3-manifold $M = E_t$. There are a variety of choices for χ. If we intend to keep track of both orientations and colors of t, then we consider the isomorphism $\chi_M : H_1M \to \Pi \cong \langle x_1, \ldots, x_d \mid \ \rangle$ that maps the class of the ith oriented meridian to x_i. In this case, $\tilde{M}$ is the universal abelian cover of M. Alternatively, we can keep track of orientations but ignore colors. Then we consider $\chi_M : H_1M \to \Pi \cong \langle x \mid \ \rangle$, mapping the class of each oriented meridian to x. The covering space $\tilde{M}$ is sometimes called the "total linking number cover."

Krebes's theorem will arise in another way, letting M be the 2-fold cyclic cover of B^3 branched over t and letting Π be the trivial group. In this case, $\tilde{M}$ is equal to M.

Having chosen a nontrivial epimorphism χ and associated cover $p : \tilde{M} \to M$, we consider the relative homology group $H_1(\tilde{M}, p^{-1}(*))$, which is a finitely generated module over $\Lambda = \mathbb{Z}\Pi$. We denote the module by $\mathcal{A}_t$ and call it the *Alexander-Fox module* of t. Generators for $\mathcal{A}_t$ can be found corresponding to the arcs of any diagram of t; a set of defining relations is obtained from crossings, as in Figure 2 below. A similar description for oriented links (0-tangles) is well known. Details can be found in [**SW00**], for example.

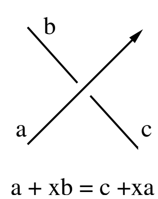

Figure 2. Crossing relation in $\mathcal{A}_t$

In the case that M is the 2-fold cyclic cover of B^3 branched over t and χ is trivial, we can still obtain $H_1M \oplus \mathbb{Z}$ using the procedure above, letting $x = -1$. Killing the generator corresponding to any single arc yields H_1M.

When $d > 1$ the Alexander module $\mathcal{A}_t$ is easier to present with generators and relators than is $H_1\tilde{M}$. The two modules fit into the following exact sequence which arises from the long exact homology sequence of the pair $p^{-1}(*) \subset \tilde{M}$.

$$0 \;\to\; H_1\tilde{M} \xrightarrow{f} \mathcal{A}_t \xrightarrow{g} \epsilon(\Lambda) \to\; 0 \tag{3.1}$$

Here $\epsilon(\Lambda)$ is the ideal of Λ generated by $x_1 - 1, \ldots, x_d - 1$. Since $\epsilon(\Lambda)$ is a submodule of Λ, it is torsion-free. It follows from exactness that $TH_1\tilde{M} \cong T\mathcal{A}_t$. Hence $\tau = \Delta_0(TH_1\tilde{M})$ can be computed as $\Delta_0(T\mathcal{A}_t)$. It is not necessary to find the torsion submodule; Lemma 2.3 (2) ensures that τ is equal to the first nonzero elementary divisor $\Delta_k(\mathcal{A}_t)$.

We can also compute the boundary invariant β using the Alexander module. Let D be the submodule of $\mathcal{A}_t$ generated by its Λ-torsion elements and by generators associated to input and output arcs of t. By Lemma 2.3 (4) the sequence

$$0 \;\to\; H_1\tilde{M}/f^{-1}(D) \xrightarrow{\bar{f}} \mathcal{A}_t/D \xrightarrow{\bar{g}} \epsilon(\Lambda)/g(D) \to 0$$

is exact. The preimage $f^{-1}(D)$ is generated by $TH_1\tilde{M}$ together with the image of $i_{1+} : H_1(\partial_+\tilde{M}) \to H_1\tilde{M}$. Hence $H_1\tilde{M}/f^{-1}(D) \cong BH_1\tilde{M}/\mathrm{im}\ \nabla$ (see Definition 2.1). Each generator is mapped by g to $x_j - 1$ in the exact sequence above, where j corresponds to the component of t to which the associated arc belongs. Hence $\epsilon(\Lambda)/g(D)$ is either trivial (for example, if t has no closed components) or else its rank is 1. In the latter case, we apply the following.

Lemma 3.3. (Lemma 7.2.7(3), [**Kw96**]) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated Λ-modules. If $TC = 0$ and the rank of C is 1, then $\Delta_0(A) \doteq \Delta_1(B)$.

Example 3.4. Consider the colored, oriented "square-tangle" t with arcs labeled as in Figure 3 below. The Alexander-Fox module $\mathcal{A}_t$ has a presentation with generators a, b, c, d, e, f, g, h and relations:

$$b + ya = c + xb,\ c + xb = d + yc,\ d + yc = e + xd,$$
$$b + hy = g + xb,\ g + xb = f + yg,\ f + yg = e + xf.$$

(Here we use x, y instead of the more cumbersome x_1, x_2.) By elementary operations we find that c, d, e, f and g can be expressed in terms of a, b and h. More precisely:

$c = ya+(1-x)b, d = (y-y^2)a+(1-y+xy)b, e = (y-xy+xy^2)a+(1-x+xy-x^2y)b, f = (1-y+xy)b+(y-y^2)h$ and $g = (1-x)b+yh$. We have

$$\mathcal{A}_t \cong \langle a, b, h \mid (1-x+xy)a = (1-x+xy)h\rangle.$$

The Λ-torsion submodule is isomorphic to $\Lambda/(1-x+xy)$. Hence $\tau = 1-x+xy$. We can compute the boundary invariant β by first killing the images of a, d, f and h in $BH_1\tilde{M} \cong \langle b \mid \;\rangle$, obtaining the quotient module $\langle b \mid (1-y+xy)b\rangle$, and then taking the 0th elementary divisor. We find that $\beta = 1-y+xy$.

If we reverse the orientation of one component of t, say the first, then τ and β become $1-x^{-1}+x^{-1}y \doteq 1-x-y$ and $1-y+x^{-1}y \doteq x+y-xy$, respectively.

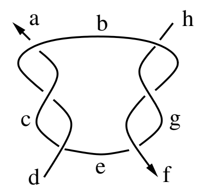

Figure 3: Labeled tangle t

Remark 3.5 In the example above β divides $\bar{\tau}$ (in fact, they are equal). In all examples that we have computed, β divides $\bar{\tau}$ provided that β is nonzero.

Assume that a $2n$-tangle t embeds in a link $\ell = \ell_1 \cup \cdots \cup \ell_d$. We denote the exterior $S^3 - int\ N(\ell)$ by E_ℓ. If colors and orientations of t (if any) match those of ℓ, then the augmentation homomorphism χ for t extends over H_1E_ℓ. We will assume that this is the case. Let $\tilde{E}_\ell$ denote the corresponding cover of E_ℓ.

If t and ℓ are colored and oriented, and χ maps the ith oriented meridian to $x_i \in \Pi$, then $\tilde{E}_\ell$ is the maximal abelian cover of ℓ; in this case, $\Delta_0(H_1\tilde{E}_\ell)$ is the multivariable Alexander polynomial $\Delta_l(x_1, \ldots, x_d)$ of the link. If ℓ is merely oriented, and χ maps each oriented meridian to $x \in \Pi \cong \langle x \mid \;\rangle$, then $\tilde{E}_\ell$ is an infinite cyclic cover; in this case, $\Delta_0(H_1\tilde{E}_\ell)$ is the 1-variable Alexander polynomial $\Delta_l(x)$ of the link. In either case, Theorem 2.2 implies that the product $\tau\beta$ of the torsion and boundary invariants of t divides the Alexander polynomial of ℓ.

The 1-variable Alexander polynomial of an oriented link is related to the multivariable Alexander polynomial. The following Lemma is a consequence of Proposition 7.3.10(1) of [**Kw96**].

Lemma 3.6. If ℓ is an oriented link of $d > 1$ components, then $\Delta_l(x) \doteq (x-1)\Delta_l(x, \cdots, x)$.

Both the multivariable and single-variable Alexander polynomials can be found directly from a diagram for ℓ. Consider the $\mathbb{Z}\Pi$-module with generators $a, b, c, \ldots$ corresponding to the arcs of the diagram and relations associated to the crossings, as in Figure 2. (Here Π is $\langle x_1, \ldots, x_d \mid \rangle$ or $\langle x \mid \rangle$, depending on which polynomial is desired.) One builds a presentation matrix with columns and rows corresponding to generators and relators, respectively. Any submatrix obtained by deleting a single row and column is an *Alexander matrix* of ℓ. Its determinant is the Alexander polynomial. Details can be found in [**Li97**], for example.

Finally, we consider the case that t is neither colored nor oriented. Let M be the 2-fold cyclic cover of B^3 branched over t. When t embeds in a link ℓ, then M embeds in N, the 2-fold cyclic cover of S^3 branched over ℓ. In order to apply Theorem 2.2 we let χ be the homomorphism mapping H_1M and H_1N to the trivial group. It is well known that $\Delta_0(H_1N)$ is the determinant of ℓ (see for example [**Li97**]). It follows from [**Ru00**] that the product $\tau\beta$ is the greatest common divisor of det $n(t)$ and det $d(t)$. Hence Krebes's theorem (Theorem 2.1) is a consequence of Theorem 2.2. For the convenience of the reader we repeat the argument of [**Ru00**].

Assume that t is an uncolored, unoriented 4-tangle that embeds in a link ℓ. Let M be the 2-fold cyclic cover of B^3 branched over t. The boundary of M is a torus. Let N be the 2-fold cyclic cover of S^3 branched over ℓ. Then $M \subset N$. Moreover, the order of H_1N is the determinant of ℓ. Poincaré duality implies that M and $M' = N - \text{im } M$ are rational homology circles. Hence we can write $H_1M \cong \mathbb{Z} \oplus \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_s$, for some positive integers $q_1, \ldots, q_s$. Note that the order of TH_1M is $|q_1 \cdots q_s|$. From the long exact homology sequence of the pair $M \subset N$:

$$\cdots \to H_2(N, M) \xrightarrow{\partial} H_1M \to H_1N \to \cdots$$

we see that $H_1M/\partial H_2(N, M)$ embeds in H_1N. By the excision isomorphism $H_2(N, M) \cong H_2(M', \partial M)$ is infinite cyclic, generated by a relative 2-cycle C. The boundary ∂C represents a class $\gamma \in H_1M$. By what we have already said, $H_1M/\langle\gamma\rangle$ embeds in H_1N. Let $i_{1+} : H_1(\partial M) \to H_1M$ be the homomorphism induced by inclusion. Then $i_{1+}C = (c, c_1, \ldots, c_s)$, for some integers $c, c_1, \ldots, c_s$. Note that the order of $H_1M/\langle\gamma\rangle$ is equal to $|c|$ times the order of TH_1M.

It is clear that $H_1M/\langle\gamma\rangle$ is presented by the matrix

$$\begin{pmatrix} c & c_1 & \cdots & c_s \\ 0 & q_1 & 0 & \cdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & q_s \end{pmatrix}.$$

The class γ is equal to $m_1\mu + m_2\lambda$, where μ is the class of the meridian of ∂M, ℓ is the class of the longitude and m_1, m_2 are integers. Certainly gcd $(m_1, m_2) \cdot |q_1 \ldots q_s|$

divides the order of H_1N. However, $|m_1 \cdot q_1 \ldots q_s|$ and $|m_2 \cdot q_1 \ldots q_s|$ are easily seen to be the orders of H_1N when ℓ is the numerator and denominator closures of t. Hence gcd $(m_1, m_2) \cdot |q_1 \ldots q_s|$ is the greatest common divisor of the determinants of $n(t)$ and $d(t)$, and it divides the determinant of ℓ.

The relation between this proof, due to Ruberman, and our approach is the following. The absolute value of the boundary invariant β is the order of $BH_1M/\mathrm{im}\ \nabla$. This quotient module of $H_1M \cong \mathbb{Z} \oplus \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_s$ can be obtained by killing the torsion elements and then killing the cosets of $i_{1+}\mu$ and $i_{1+}\lambda$. The result is a cyclic group of order gcd (m_1, m_2). Hence gcd $(m_1, m_2) = |\beta|$. The quantity $|q_1 \cdots q_s|$ is the order of TH_1M, which is $|\tau|$.

Example 3.7. We return to the square tangle t of Example 3.4. Ignoring colors we find that t and β are both equal to $x^2 - x + 1$. Hence $(x^2 - x + 1)^2$ divides the Alexander polynomial of any oriented link in which t embeds.

If we ignore the orientation of t as well, then τ and β are both equal (up to sign) to 3. Hence 9 divides the determinant of any link ℓ in which t embeds.

Example 3.8. Consider the uncolored, oriented 6-tangle t in Figure 4. In the exact sequence (3.1) the module $\epsilon(\Lambda)$ is cyclic, and thus the sequence splits. As a result $H_1\tilde{M}$ is isomorphic to the quotient $\mathcal{A}_t^0$ of the Alexander-Fox module $\mathcal{A}_t$ obtained by killing a single meridianal generator. In Figure 4, five meridianal generators are labeled, one of them with zero; the remaining generators can be written in terms of these. A presentation matrix A can be found for $\mathcal{A}_t^0$. For convenience we choose A to be square (4×4) with two zero rows. We used the software package *Maple*, which enabled us to find nonsingular 4×4 matrices U, V over the ring $R = \mathbb{Q}\Pi$ such that UAV is a diagonal matrix $\mathrm{diag}(1, \tau(x), 0, 0)$, the Smith normal form of A, where $\tau(x) = (x^2 - 4x + 1)(x^2 - x + 1)(x - 1)$. Hence

$$\mathcal{A}_t^0 \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Q}\Pi)^2 \oplus \mathbb{Q}\Pi/(\tau(x)).$$

The polynomial $\tau(x)$ is the torsion invariant of the 6-tangle. In order to find the boundary invariant $\beta(x)$ we project the five generators of $\mathcal{A}_t^0$ corresponding to input/output arcs of t onto the free part of $\mathcal{A}_t^0 \otimes_{\mathbb{Z}} \mathbb{Q}$, using the matrix V. The images comprise the rows of a 5×2-matrix, and the greatest common divisor of the the 2×2-minors of this matrix is $\beta(x)$. We find that $\beta(x) = x^2 - x + 1$. By Theorem 3.5 the polynomial $\beta\tau = (x^2 - x + 1)^2(x^2 - 4x + 1)(x - 1)$ divides the Alexander polynomial of any oriented link ℓ in which t embeds. Ignoring orientations, we find that 108 divides the determinant of ℓ.

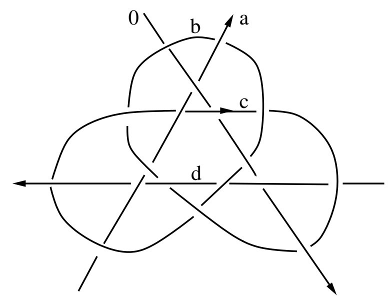

Figure 4: Labeled 6-tangle t

4. Virtual links and $2n$-tangles. In 1996 L. Kauffman introduced the notion of a virtual link [**Ka97**], thereby extending the "classical" category of links in an interesting and nontrivial manner. We review the main ideas. The reader can find additional information in [**Ka99**] or [**Ka00**].

A diagram for a classical link is a a planar 4-regular graph with information at each vertex indicating how the link crosses itself when viewed from a fixed perspective. The decorated vertex is called a crossing. A *virtual link diagram* is likewise defined. However, such a diagarm is permitted to contain crossings of a new, "virtual" type. Classical and virtual crossing conventions appear in Figure 5. In many respects virtual crossings are treated as though they are not present. For example, the arcs of a diagarm are defined to be the maximal connected components, just as for classical link diagrams, regardless of the virtual crossings that they might contain.

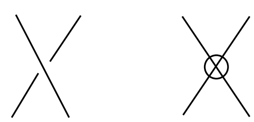

Figure 5: Classical and virtual crossings

Two virtual link diagrams are equivalent if one can be obtained from the other by a finite sequence of the usual, classical Reidemeister moves or "virtual" Reidemeister moves. Virtual Reidemeister moves are shown in Figure 6. By a generalized Reidemeister move we mean either a classical or virtual Reidemeister move.

A *virtual link* is an equivalence class of diagrams. We define *virtual $2n$-tangle diagram* and *virtual $2n$-tangle* in the same manner. Orientations can be imposed as in the classical category. As before, a virtual tangle means a virtual 4-tangle. The numerator closure and the denominator closure are defined as in the classical case.

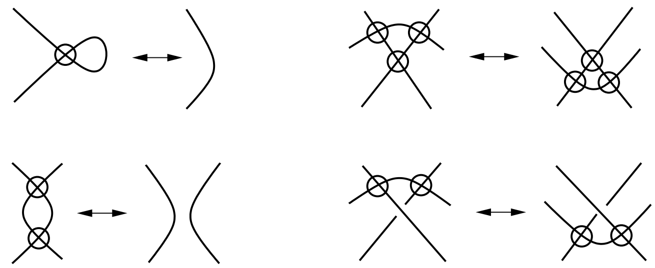

Figure 6: Virtual Reidemeister moves

A large body of virtual knot literature has already appeared. One reason for the interest is that virtual knot theory contains the classical theory; more precisely, if two classical knots are equivalent under generalized Reidemeister moves, then they are equivalent under classical ones. A proof can be found in [**GPV00**] (see also [**Ka**]).

An Alexander matrix can be associated to a diagram of an oriented virtual link ℓ by the same procedure as in section 3 (see the paragraph following the proof of Lemma 3.4.) The *Alexander polynomial* $\Delta_\ell(x)$ is defined to be the greatest common divisor of the determinants of all submatrices obtained by deleting a single row and column.

When ℓ is a classical link any row of an Alexander matrix is a linear combination of the other rows, and consequently the determinants of any two submatrices differ by a unit factor. This need no longer be true when ℓ is virtual, and in that case the determinants of *all* the submatrices must be considered.

Setting all of the variables equal to -1 in the Alexander matrix matrix, and then taking the greatest common divisor of the submatrices produces an integer. Its absolute value is called the *determinant* $\det(\ell)$ of the link. It is well known that in the classical case, $\det \ell$ is equal to the absolute value of the Kauffman bracket polynomial of ℓ evaluated at a primitive eighth root of unity ζ. However, for virtual knots and link, such an evaluation might not agree with the determinant; in fact, it need not be an integer. (This anomaly was pointed out in [**SW99**].)

A virtual $2n$-tangle t *embeds* in a virtual link ℓ if some diagram for t extends to a diagram for ℓ.

Theorem 4.1. [**SW99**] Assume that t is a virtual tangle that embeds in a virtual link ℓ. If d is a square-free integer that divides the determinants of both the numerator closure and the denominator closure of t, then d divides $\det(\ell)$.

We give an example to show that the hypothesis that d is square-free cannot be relaxed.

Example 4.2. Figure 7 shows the square-tangle t embedded in a virtual link ℓ. Recall that $\det(n(t)) = 0$ while $\det(d(t)) = 9$. An elementary calculation shows that $\det(\ell) = 3$.

Hence the conclusion of Theorem 4.1 does not hold when d is 9. However, the evaluation of the Kauffman bracket polynomial of ℓ at a primitive eighth root of unit is equal to 9. We will return to this example in Section 5 (see Example 5.6).

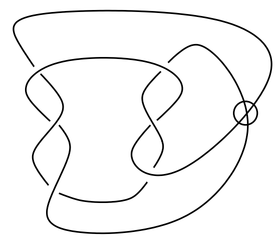

Figure 7: Embedded square-tangle

5. Persistent invariants from skein theory. Skein theory and related ideas provide obstructions to embedding tangles. First we discuss applications of the Kauffman bracket skein theory, which allows us to generalize Krebes's theorem to $2n$-tangles, for any n. Our knot and link notation follows [**Ro76**].

A *Catalan tangle* is a $2n$-tangle without crossings or trivial components. There are $\frac{1}{n+1}\binom{2n}{n}$ Catalan tangles. If t and s are $2n$-tangles, then t^s denotes the link obtained by closing t by s; that is, by joining the corresponding ends of t and s without introducing crossings. If t is a 4-tangle and s is the 0-tangle (respectively ∞-tangle) as in Figure 8, then t^s is the numerator closure (respectively, denominator closure) of t. Finally $\langle \ell \rangle$ denotes the Kauffman bracket polynomial of a framed link $\ell \subset S^3$. The reader might consult [**Ka91**] for background.

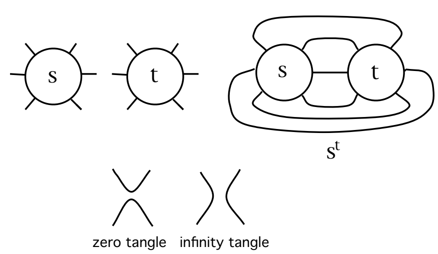

Figure 8: $2n$-tangle closure s^t, 0-tangle and ∞-tangle

The main result involving the Kauffman bracket polynomial is the following. It offers a technique for deciding whether a $2n$-tangle embeds in a link.

Theorem 5.1. If a $2n$-tangle t embeds in a link ℓ, then the ideal $\mathcal{I}_t$ of $\mathbb{Z}[A^{\pm 1}]$ generated by Kauffman bracket polynomials of all diagrams $\langle t^c \rangle$, where c is any Catalan tangle, contains the polynomial $\langle \ell \rangle$.

Proof. Assume that ℓ is of the form t^s. We use the Kauffman bracket skein relation[1] to eliminate all crossings of s, and then eliminate all trivial components of s. The resulting links are of the form t^c, where c are Catalan tangles. As a consequence, $\langle \ell \rangle$ is a linear combination of $\langle t^c \rangle$ with the coefficients in $\mathbb{Z}[A^{\pm 1}]$. ∎

We reformulate Theorem 5.1 in the language of Jones polynomials, recalling that Jones polynomials of the same link with various orientations differ only by multiplication by units in $\mathbb{Z}[t^{\pm 1/2}]$. We also use the fact that the determinant of the link $\det(\ell)$ is the absolute value of the Jones polynomial evaluated at $t = -1$.

Corollary 5.2 (i) If a $2n$-tangle t embeds in a link ℓ, then the ideal of $\mathbb{Z}[t^{\pm 1/2}]$ generated by Jones polynomials V_{t^c}, where c is any Catalan tangle, contains $V_\ell(t)$.
(ii) The greatest common divisor of the determinants of t^c, where t ranges over all Catalan closures c, divides the determinant of ℓ.

We illustrate the usefulness of Theorem 5.1 by analyzing the tangle t in Figure 9, and considering possible links in which it embeds.

Example 5.3. The tangle t in Figure 9 appears in [**Kr99**].

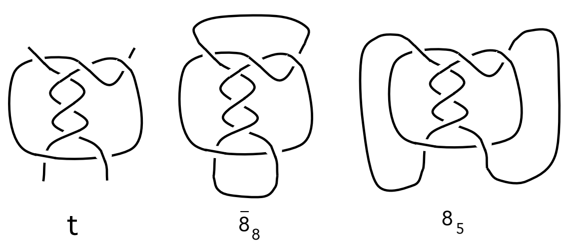

Figure 9: Krebes's tangle and closures

The ideal $\mathcal{I}_t$ is $(\langle \bar{8}_8 \rangle, \langle 8_5 \rangle) = (17, A^4 - 5)$. It is a proper ideal that does not contain $(A^2 + A^{-2})^n$, for any positive integer n, nor does it contain $\langle 4_1^2 \rangle = -A^{10} + A^6 - A^2 - A^{-6}$. Hence t does not embed in the Hopf link, any trivial link or the link 4_1^2. Furthermore, one can check that for knots up to 8 crossings, the polynomial $\langle k \rangle$ is contained in the ideal $(17, A^4 - 5)$ only when k is 6_2, $\bar{8}_1$, $\bar{8}_{14}$, or of course 8_5 and $\bar{8}_8$. In order to exclude 6_2 and $\bar{8}_1$, we use the criterion based on the Homflypt polynomial (Theorem 5.7). Similarly, one excludes $\bar{8}_{14}$ with central strands oriented in the same direction.

To find the ideal $\mathcal{I}_t$ there is no need to consider Catalan tangles as in Theorem 5.1. We used Catalan tangles because they form a natural basis of the Kauffman bracket

[1] The relations are $\langle \ell_+ \rangle = A\langle \ell_0 \rangle + A^{-1}\langle \ell_\infty \rangle$ and $\langle \ell \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle \ell \rangle$.

skein module $\mathcal{S}_{2,\infty}$ of a $2n$-tangle [**Pr91**],[**Pr99**]. Instead we can use any family of $2n$-tangles that generate the skein module, which often allows us to shorten the computation significantly. In the case of Krebes's tangle t, we can replace the numerator $n(t)$ with the tangle $s =$ . Then t^s is the $(4,2)$-torus link $\bar{4}_1^2$, with Kauffman bracket polynomial $-A^{-10}+A^{-6}-A^{-2}-A^6$. This polynomial is simpler than that of $n(t)$. We have $\mathcal{I}_t = (\langle \bar{4}_1^2 \rangle, \langle 8_5 \rangle) = (-A^{-10}+A^{-6}-A^{-2}-A^6,\ A^{12}-A^8+3A^4-3+3A^{-4}-4A^{-8}+3A^{-12}-2A^{-16}+A^{-20}) = (17, A^4-5)$.

Theorem 5.1 generalizes in several directions. For example, we can ask when one tangle embeds in another tangle, as illustrated by Theorem 5.4. The proof is similar to the proof of Theorem 5.1. The product $s \cdot t$ of $2n$-tangles is defined as usual by placing the diagram for t to the right side of a diagram for s, and then connecting arcs, as in Figure 10.

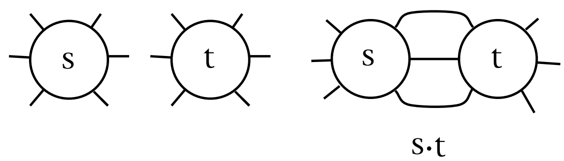

Figure 10: $2n$-tangle product

Theorem 5.4. Let s and t be $2n$-tangles. If there exists a $2n$-tangle u such that t is the composition $s \cdot u$, then the element of the Kauffman bracket skein module[2] represented by $s \cdot u$ is contained in the submodule generated by the elements $s \cdot c$, where c ranges over all Catalan $2n$-tangles.

Another direction extends Theorem 5.1 to the virtual category. When $n > 2$, the ideal associated to a $2n$-tangle may be larger than in the classical case, as we will illustrate later. The proof of Theorem 5.5 is similar to the proof of Theorem 5.1: we simplify as much as possible the complement $2n$-tangle $t' = \ell - t$ using Kauffman bracket relations extended to the virtual category.

Theorem 5.5. Let t be a virtual $2n$-tangle and ℓ a virtual link. Let $\mathcal{I}_t^v$ be the ideal of $\mathbb{Z}[A^{\pm 1}]$ generated by Kauffman bracket polynomials of closures t^v, where v ranges over the $(2n)!/2^n n!$ virtual $2n$-tangles corresponding the various ways that the $2n$ boundary

[2] The Kauffman bracket skein module of a 3-ball with $2n$ points on its boundary is the quotient of the free module generated by framed unoriented $2n$-tangles modulo the submodule generated by elements $\ell_+ - A\ell_0 - A^{-1}\ell_\infty$ and $\bigcirc \cup \ell + (A^2 + A^{-2})\ell$.

points can be connected by n arcs without classical crossings. If t embeds in ℓ, then $\langle \ell \rangle$ is contained in $\mathcal{I}_t^v$.

Example 5.6. Consider the tangle t in Figure 3. The ideal $\mathcal{I}_t = (A^4+1, (A^{12}+A^4-1)(A^{12}-A^8-1))$ is equal to $(A^4+1, 9)$. The bracket polynomial of the virtual closure t^v is $A^{20}+A^{18}-A^{16}-2A^{14}+3A^{10}-2A^6-A^4+A^2+1$. Since this polynomial is contained in $\mathcal{I}_t$, the ideals $\mathcal{I}_t$ and $\mathcal{I}_t^v$ are the same in this case. The single nonclassical closure t^v appears in Figure 7.

Setting A equal to a primitive eighth root of unity ζ, reduces $\mathcal{I}_t^v$ to the ideal (9) of the ring $\mathbb{Z}[\zeta]$. Hence for any virtual link ℓ in which t embeds, the absolute value of $\langle \ell \rangle$ evaluated at ζ must be divisible by 9 in $\mathbb{Z}[\zeta]$. (Compare with Example 4.2. See comments preceding Theorem 4.1.)

Example 5.6 illustrates a general result:

Proposition 5.7. For any classical 4-tangle t, the ideals $\mathcal{I}_t$ and $\mathcal{I}_t^v$ are equal.

Proof. If a link diagram D has exactly one virtual crossing, denoted by v, then

$$(d+1)\langle D_v \rangle = \langle D_0 \rangle + \langle D_\infty \rangle,$$

where $d = -A^2 - A^{-2}$. In order to see this, we use Kauffman bracket relations to eliminate all classical crossings and all trivial components not involving v. Since v is the only virtual crossing, it suffices to consider only diagrams that are numerator and denominator of the 4-tangle composed of a single virtual crossing. For such D, the skein relation clearly holds.

We complete the proof of the proposition by applying the above observation to the nonclassical closure t^v. We have $(d+1)\langle t^v \rangle \in \mathcal{I}_t = (\langle n(t) \rangle, \langle d(t) \rangle)$. We argue that $\langle t^v \rangle \in \mathcal{I}_t$. For this it suffices to show that $(d+1, \mathcal{I}_t) = (1)$ since then $(d+1)\langle t^v \rangle \in \mathcal{I}_t$ is equivalent to $\langle t^v \rangle \in \mathcal{I}_t$.

Since $d+1$ divides A^8+A^4+1, the ideal $(A^8+A^4+1, \mathcal{I}_t)$ is contained in $(d+1, \mathcal{I}_t)$. Notice that $\mathcal{I}_t$ contains the bracket polynomial of some classical link ℓ with an odd number of components. It follows from statement 12.4 of [**Jo87**] that $(-A^{-3})^w \langle \ell \rangle - 1$ is divisible by A^8+A^4+1, where w is the sum of the signs of the crossings of a diagram for ℓ used to compute $\langle \ell \rangle$. Consequently, $(A^8+A^4+1, \mathcal{I}_t) = (1)$ and hence $(d+1, \mathcal{I}_t) = (1)$. ■

Next we generalize Theorem 5.1 by using Homflypt and Kauffman polynomials in place of the Kauffman bracket polynomial. The choice of ring is important.

The skein relation of the Homflypt polynomial is

$$v^{-1} P_{\ell_+}(v,z) - v P_{\ell_-}(v,z) = z P_{\ell_0}(v,z).$$

For the initial data we take $P_{U_n} = (\frac{v^{-1}-v}{z})^{n-1}$, where U_n denotes the trivial link of n components. One might consider the value of the invariant in the ring $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, but then every ideal containing z would coincide with the ring. It is better to use the smaller ring $\mathcal{R} \subset \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ generated by $v^{\pm 1}, z$ and $\frac{v^{-1}-v}{z}$. This point of view is used, for example, in [**Pr89**]. The following result is easily proved.

Theorem 5.8. If an oriented $2n$-tangle t embeds in an oriented link ℓ, then P_ℓ is contained in the ideal of $\mathcal{R}$ generated by polynomials P_{t^h}, where h ranges over the $n!$ oriented $2n$-tangles that generate the Homflypt skein module.

We have taken the ring $\mathcal{R}$ instead of a more familiar ring $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ in order to get a stronger result. For example, the reduction of the ring modulo z^k leading to Vassiliev invariants is now possible (compare [**Pr94**]).

The question of whether a given element is in an ideal of a polynomial ring can be decided algorithmically using Gröbner bases, provided that the coefficient ring is a principal ideal domain. It can be applied to Krebes's tangle in Example 5.3, for example, by using the Homflypt polynomial and computing the Gröbner basis of the associated ideal in $\mathcal{R} = \mathbb{Z}[v, w, z, y]/(vw-1, zy-v^{-1}+v)$. As another example, the ideal $\mathcal{J}_t$ for the oriented tangle t in Figure 9 with strands of the central 3-twist oriented in the same direction is generated by $P_{\bar{4}_1^2}$ and P_{8_5}. From the form of its Gröbner basis in $\mathcal{R}$ we can conclude that $P_{6_2}, P_{\bar{8}_1}$ and $P_{8_{14}}$ are not elements of the ideal. We are grateful to M. Dąbkowski for computations in the ring $\mathcal{R} \otimes \mathbb{Z}/17\mathbb{Z}$ performed with the program GAP.

Corollary 5.9. The Alexander-Conway polynomial $\nabla_\ell(z)$ is contained in the ideal of $\mathbb{Z}[z^{\pm 1}]$ generated by elements $\nabla_{t^h}(z)$.

Corollary 5.9 implies that if the 4-tangle t can be embedded in a link ℓ, then ∇_ℓ is in the ideal generated by two polynomials, Alexander-Conway polynomials of the two links obtained from t by closing with its ends using fewest possible crossings. When the orientations of the boundary arcs alternate as one travels along the perimeter of a diagram, these are the numerator and denominator closures; otherwise, one closure acquires an additional crossing while the other does not.

Corollary 5.9 is useful, as we have seen in Example 3.4. However, it many cases embedding criteria based on the Homflypt polynomial is more helpful. For example, if we try to apply Corollary 5.9 to Krebes's tangle (Figure 9), which we have seen does not embed in many links, we find that the ideal generated by $\nabla_{n(t)}$ and $\nabla_{d(t)}$ is trivial, regardless of the orientations chosen.

Similarly we have an obstruction for embedding an unoriented tangle in a link using the 2-variable Kauffman polynomial. We use the fact that the Kauffman polynomials of links that differ only by orientation of their components are the same up to multiplication by a unit in the ring. As before, the ring is taken to be $\mathcal{R}$.

Theorem 5.10. If an unoriented $2n$-tangle t embeds in an unoriented link ℓ, then Λ_ℓ is contained in the ideal of $\mathcal{R}$ generated by Λ_{t^κ}, where κ ranges over the $\frac{(2n)!}{2^n n!}$ elements that generate the Kauffman skein module[3] of $2n$-tangles [**Pr91**].

We can look at our criteria from the more general point of view of bilinear forms on skein modules. Let $\mathcal{S}(B^3, 2n)$ be a skein module of B^3 with $2n$ points on its boundary. (See [**HP92**] or [**Pr2**] for details.) It is the quotient of a free module generated by $2n$-tangles by the submodule generated by properly chosen skein expressions. We have many choices for skein relations. There is a bilinear form $\phi : \mathcal{S}(B^3, 2n) \times \mathcal{S}(B^3, 2n) \to \mathcal{S}(S^3)$ in which pairs of $2n$-tangles are joined together to form links. The skein module $\mathcal{S}(S^3)$ is a ring in which product is defined by distant union. We obtain the following embedding criterion. If t embeds in ℓ, then ℓ is in the image of the restriction $\phi(t, \ )$. The image is the submodule spanned by elements $\phi(t, g)$, where g ranges over a generating set for $\mathcal{S}(B^3, 2n)$. For computational reasons we prefer $\mathcal{S}(B^3, 2n)$ to be finitely generated as is the case for Kauffman bracket, Homflypt and Kauffman skein modules.

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